Lecture 3:

Definition:(Block circulant)

V is block-circulant $\Longleftrightarrow$ $V = \begin{pmatrix} H_0^{N\times N} & H_{n-1} & \cdots & H_1 \\ H_1^{N\times N} & H_0 & & H_2 \\ \vdots & \vdots & \cdots & \vdots \\ & & & H_{n-1} \\ H_{n-1}^{N\times N} & H_{n-2} & \cdots & H_0 \end{pmatrix} \ni$

each H_i is a circulant matrix.

Theorem: If H = transf. matrix of shift-invariant operator,

then $H = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ A_{21} & \cdots & A_{2N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$ where each A_{ij} is a circulant matrix.

Theorem: H is block circulant!

(Exercise)

Proof:

Consider
$$A_{ij} = \begin{array}{cc} & x \rightarrow \\ \begin{array}{c}\alpha \\ \downarrow\end{array} & \left(\begin{array}{c} y = j \\ \beta = i \end{array} \right) \end{array}$$

$$\therefore A_{ij} = \begin{pmatrix} h(1,1,j,i) & h(2,1,j,i) & \cdots & h(N,1,j,i) \\ h(1,2,j,i) & h(2,2,j,i) & \cdots & h(N,2,j,i) \\ \vdots & \vdots & & \vdots \\ h(1,N,j,i) & h(2,N,j,i) & \cdots & h(N,N,j,i) \end{pmatrix}$$

Shift-invariant $\Leftrightarrow$ $h(x,\alpha,y,\beta) = g(\alpha - x, \beta - y)$ for some g.

$$\therefore A_{ij} = \begin{pmatrix} g(0, i-j) & g(\cancelto{N-1}{-1}, i-j) & \cdots & g(\cancelto{1}{1-N}, i-j) \\ h(1, i-j) & g(0, i-j) & \cdots & g(\cancelto{2}{2-N}, i-j) \\ \vdots & \vdots & & \vdots \\ h(N-1, i-j) & g(N-2, i-j) & \cdots & g(0, i-j) \end{pmatrix} \leftarrow \text{Circulant}$$

(Assume periodic property)

Properties of separable image transformation

Recall: Separable $h \Leftrightarrow h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$.

Let $\vec{g} = H\vec{f}$.

H: transformation matrix

$$\Longrightarrow g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha) \underbrace{\sum_{y=1}^{N} f(x,y)\, h_r(y,\beta)}_{\text{Matrix multiplication}}$$

Consider $h_r = \left(h_r(y,\beta)\right)_{1\le y,\beta\le N} \in M_{N\times N}$

$h_c = \left(h_c(x,\alpha)\right)_{1\le x,\alpha\le N} \in M_{N\times N}$ Let $S = f\, h_r$.

$f = \left(f(x,y)\right)_{1\le x,y\le N} \in M_{N\times N}$

Easy to see: $g(\alpha,\beta) = \sum_{x=1}^{N} h_c(x,\alpha)\, S(x,\beta) = \sum_{x=1}^{N} h_c^T(\alpha,x)\, S(x,\beta)$

$\therefore\ g = h_c^T S = h_c^T f\, h_r$ (Matrix form)

Definition: (Kronecker product)

Let A and B be two matrices.

Kronecker product of A and B = $\underset{\substack{\| \\ (a_{ij})_{1 \le i,j \le N}}}{A} \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1N}B \\ a_{21}B & \cdots & a_{2N}B \\ \vdots & & \vdots \\ a_{N1}B & \cdots & a_{NN}B \end{pmatrix}$

Theorem: Consider a separable $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$.

Then its transformation matrix H is:

$$h_r^T \otimes h_c^T$$

Exercise!

Image decomposition

Suppose $h(x,\alpha,y,\beta) = h_c(x,\alpha)\, h_r(y,\beta)$ (Separable).

Then: $g = h_c^T f h_r \Rightarrow f = (h_c^T)^{-1} g (h_r)^{-1}$

Write: $(h_c^T)^{-1} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$; $h_r^{-1} = \begin{pmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ & \vdots & \\ - & \vec{v}_N^T & - \end{pmatrix}$

Then: $f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij}\, \vec{u}_i \vec{v}_j^T$ $\leftarrow M_{N\times N}$

Check that: $(h_c^T)^{-1} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} h_r^{-1} = \vec{u}_i \vec{v}_j^T$

(i,j)-entry

$\therefore f =$ linear combination of $\{\vec{u}_i \vec{v}_j^T\}_{i,j}$

Definition: Each $\vec{u}_i v_j^T$ is called an elementary image.
$\vec{u}_i v_j^T$ is also called the outer product of $\vec{u}_i$ and $\vec{v}_j$.

One important task in image processing:

Choose h_c and h_r such that:

1. Transformed image requires less storage (Many $g_{ij} = 0$)
2. Take away some terms $g_{ij}\, \vec{u}_i \vec{v}_j^T$ (e.g. high-frequency) $\rightarrow$ Better image!!
3. h_c^{-1} and h_r^{-1} are easy to compute!

Common example: Unitary matrices.

Unitary $U \Leftrightarrow UU^* = I$ (U^* = conjugate transpose)

$$= \begin{cases} (\bar{U})^T & \text{if } U \text{ is complex} \\ U^T & \text{if } U \text{ is real.} \end{cases}$$

Stacking operator

Definition: (Stacking operator)

Define: $\vec{V}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \\ \leftarrow \text{row } n \\ \\ \\ \\ \end{array}$ and $N_n = \begin{pmatrix} 0 \\ I_n \\ 0 \end{pmatrix} \begin{array}{l} \leftarrow (n-1)N \times N \text{ zero matrix} \\ \leftarrow N \times N \text{ identity matrix} \\ \leftarrow (N-n)N \times N \text{ zero matrix} \end{array}$

Let $f \in \mathcal{I}$. The <u>stacking operator</u> $\mathcal{S}$ on f is defined as:

$$\mathcal{S}f := \vec{f} := \sum_{n=1}^{N} N_n f \vec{V}_n$$

Remark: 1. $\mathcal{S}f \in M_{N^2 \times 1}$

2. The 1^{st} col of f forms the first N entries of $\mathcal{S}f$

The 2^{nd} col of f forms the next N entries of $\mathcal{S}f$ etc ...

Theorem: $\mathcal{S}$ is linear and $f = \sum_{n=1}^{N} N_n^T \vec{f} \, \vec{V}_n^T$

(exercise)

Similarity between images

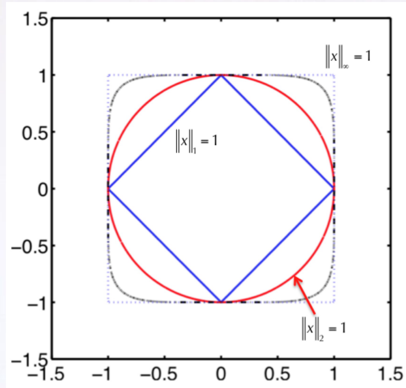
Need to define matrix norm $\|\cdot\|$ such that: for $\forall f, g \in \mathcal{I}$, we can define similarity between f and g as $\|f-g\|$.

Definition: A vector / matrix norm is a function $\|\cdot\| : \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) $\to \mathbb{R}$ so that for any $\vec{x}, \vec{y} \in \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) and $\alpha \in \mathbb{R}$, we have:

1. $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0$ iff $\vec{x} = 0$.
2. $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality)
3. $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$

Example:

$$\left.\begin{array}{ll} \bullet\ \|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i| \qquad \vec{x} = (x_1, x_2, \dots, x_m)^T \\ \bullet\ \|\vec{x}\|_2 = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2} \\ \bullet\ \|\vec{x}\|_\infty = \max_{i=1,2,\dots,m} |x_i| \end{array}\right\} \text{Vector norm}$$



Definition: (induced matrix norm) Let $A \in \mathbb{R}^{m \times m}$. We define the induced matrix norm induced by a vector norm $\| \|_v$ to be the smallest $C \in \mathbb{R}$ such that

$$\| A x \|_v \le C \|x\|_v \text{ for } \forall \vec{x} \in \mathbb{R}^m.$$

Equivalently,
$$\|A\| = \sup_{\vec{x} \in \mathbb{R}^m, \vec{x} \neq 0} \frac{\|A\vec{x}\|_v}{\|\vec{x}\|_v} = \sup_{\substack{\vec{x} \in \mathbb{R}^m \\ \|\vec{x}\|_v = 1}} \|A\vec{x}\|_v.$$

Notation: We denote the matrix norm induced by the vector norm $\| \cdot \|_p$ by the same symbol $\| \cdot \|_p$

Another commonly used matrix norm

Definition: (Frobenius norm) $\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$

Let $\vec{a}_j$ = j-th col of A. We have: $\|A\|_F = \sqrt{\sum_{j=1}^{n} \|a_j\|_2^2} = \sqrt{tr(A^* A)} = \sqrt{tr(AA^*)}$
where $tr(\cdot)$ = trace of the matrix.

Theorem: The matrix 2-norm and Frobenius norm (F-norm) are invariant under multiplication by unitary matrices).

That is, for any $A \in \mathbb{R}^{m \times n}$, and unitary matrix $U \in \mathbb{R}^{m \times m}$, we have:

$$\|UA\|_2 = \|A\|_2 \quad \text{and} \quad \|UA\|_F = \|A\|_F.$$

Proof: $\|UA\vec{x}\|_2^2 = (UA\vec{x})^T(UA\vec{x}) = \vec{x}^T A^T U^T U A \vec{x}^T = \vec{x}^T A^T A \vec{x} = \|A\vec{x}\|_2^2$

$$\therefore \|UA\|_2 = \sup_{\|\vec{x}\|_2 = 1} \|UA\vec{x}\|_2 = \sup_{\|\vec{x}\|=1} \|A\vec{x}\|_2 = \|A\|_2$$

Also, $\|UA\|_F = \sqrt{tr((UA)^T(UA))} = \sqrt{tr(A^T U^T U A)} = \sqrt{tr(A^T A)} = \|A\|_F$

Image decomposition

Image decomposition based on Singular Value Decomposition (SVD)

Definition: (SVD) For any $g \in \mathbb{R}^{m \times n}$, the singular value decomposition (SVD) of g is a matrix factorization: $g = U \Sigma V^T$, where $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ are unitary, Σ is a diagonal matrix ($\Sigma_{ij} = 0$ if $i \neq j$) with diagonal entries given by: $\underbrace{\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r}_{\text{singular values}} > 0$ with $r \leq \min(m, n)$.

Theorem: The rank of g is given by the number of non-zero singular values.

Proof: Rank = dim of column space.

Recall that $\text{rank}(AB) = \text{rank}(B)$ if A is invertible

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Suppose $g = U \Sigma V^T$. Since U and V are invertible, $\text{rank}(g) = \text{rank}(\Sigma)$

$= \#$ of non-zero singular values

Remark: Consider an image g. Let $g = U \Sigma V^T$ be the SVD of g (with diagonal entries of Σ given by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$)

1. Note that $$g = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i \, U \begin{array}{cc} \text{ith} & \\ \downarrow & \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} & \leftarrow \text{ith} \end{array} V^T = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$
   $\vec{u}_i \vec{v}_i^T$ is called the eigen-image of g under SVD.

2. For $N \times N$ image, the required storage is:
   $$(\underbrace{N}_{\vec{u}_i} + \underbrace{N}_{\vec{v}_i} + \underbrace{1}_{\sigma_i}) \times \underbrace{r}_{r \text{ terms}} = (2N+1)r$$